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A Duality Theorem for Periodic Solutions of a Class of Second Order Evolution Equations[1]

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1. INTRODUCTION

$$\ddot{u} + c\dot{u} + \ell u = f(t, u), \qquad u \in V, \qquad \dot{u} \in H. \tag{1.1}$$

Here $c > 0$ is a fixed constant, ℓ is a self-adjoint linear operator which is densely defined on H and f is a nonlinear function that is T-periodic with respect to time. We shall impose conditions on ℓ and f modeled after the semilinear telegraph equation.

The periodic problem associated to equations of this type has been studied using several methods (see [19], [12], [5], [3], [21], [2], [9] and the references there). In this paper we shall prove that two well known approaches to the study of the periodic problem for (1.1) are equivalent.

The first approach is based on the Poincaré operator. To define it we start with the Cauchy problem for (1.1), which is well posed under appropriate conditions on ℓ and f. Let $u(t; u_0, v_0)$ be the solution of (1.1) satisfying

$$u(0) = u_0 \in V, \qquad \dot{u}(0) = v_0 \in H.$$

The periodic problem for (1.1) is reduced to the study of the fixed points of the Poincaré operator

$$\mathscr{P}\colon (u_0, v_0) \to (u(T; u_0, v_0), \dot{u}(T, u_0, v_0)).$$

Typically this operator is not completely continuous and so the Leray–Schauder theory is not directly applicable. However, the dissipative

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character of the equation is sufficient to guarantee that $\mathscr{P}$ is an α-contraction. This makes possible to apply the theory developed by Nussbaum in [16] to define the degree

$$\deg(I-\mathscr{P}, G; V\times H),$$

where G is an open and bounded subset of $V\times H$.

The second approach is based on the method of Green's functions and consists in rewriting the periodic problem as an abstract integral equation. To do this we impose assumptions on ℓ implying that the linear periodic problem

$$\ddot{u}+c\dot{u}+\ell u=p(t), \qquad u(0)=u(T), \qquad \dot{u}(0)=\dot{u}(T), \qquad p\in C([0,T],H),$$

has a unique solution $u=\mathscr{G}_T p$. This solution belongs to the space of functions

$$\mathfrak{M}_T=\{u\in C^1([0,T],H)\cap C([0,T],V)/u(0)=u(T), \qquad \dot{u}(0)=\dot{u}(T)\}.$$

The nonlinear periodic problem is now equivalent to the fixed point equation

$$u=\mathscr{G}_T\mathscr{N}_T u, \qquad u\in\mathfrak{M}_T,$$

where $\mathscr{N}_T$ is the substitution operator associated to f. The operator $\mathscr{F}_T=\mathscr{G}_T\mathscr{N}_T$ is completely continuous and Leray–Schauder theory can be applied to define the degree

$$\deg(I-\mathscr{F}_T, \Omega_T; \mathfrak{M}_T),$$

where Ω_T is an open and bounded subset of $\mathfrak{M}_T$.

Let us now assume that the sets G and Ω_T have a common core; this means that a periodic solution of (1.1) belongs to Ω_T if and only if its initial conditions lie on G. The duality theorem will say that the two degrees coincide; that is,

$$\deg(I-\mathscr{P}, G; V\times H)=\deg(I-\mathscr{F}_T, \Omega_T; \mathfrak{M}_T).$$

The analogous result for ordinary differential equations is classical and a proof can be found in the book by Krasnoselskii and Zabreiko [11]. The basic idea of the proof in [11] is to connect the operators $\mathscr{P}$ and $\mathscr{F}_T$ by a chain of ingenious homotopies. In our proof we follow the ideas of [11] and adapt them to an infinite-dimensional setting. The main difficulty in this adaption comes from a well known fact: the degree cannot be defined for arbitrary continuous maps in infinite dimensions. We shall prove that the successive homotopies always remain in an admissible class where the degree is well defined.

Besides their intrinsic interest, Duality Theorems can be useful in applications. We shall combine the duality theorem with the techniques introduced in [18] to study the forced sine-Gordon equation with periodic boundary conditions. This is the equation

$$u_{tt} + cu_t - u_{xx} + a \sin u = p(t, x), \qquad u(t, x + 2\pi) = u(t, x),$$

where $a > 0$ and p is doubly periodic, and we shall obtain results on multiplicity and instability of periodic solutions. These results can be seen as partial extensions of well known results for the forced pendulum equation (see [13], [17]).

The rest of the paper is organized in six sections. Section 2 is devoted to recall some known facts in degree theory that will be employed later. In Section 3 we discuss the linear equation associated to ℓ and obtain some preliminary results. Section 4 deals with the Cauchy problem for (1.1) and presents some preliminary results on the nonlinear equation. Section 5 is dedicated to describe in rigorous terms the Poincaré operator and the Functional-Analytic approach in the study of the periodic boundary value problem. In Section 6 we state and prove the Duality Theorem. This section also contains a discussion on the value of the index of an asymptotically stable periodic solution. This index can be computed using the method of the Poincaré operator and the asymptotic fixed point theorem for α-contractions (see [15], [8]). Section 7 applies the results of the previous sections to the sine-Gordon equation.

2. MAPPINGS LIKE α-CONTRACTIONS AND DEGREE THEORY

Let X be a Banach space with norm $|\cdot|$. Given a bounded set A in X, the measure of noncompactness of A is defined as

$$\gamma(A) := \inf\{d > 0 / A \text{ can be covered by a finite number of sets with diameter less or equal than } d\}.$$

The properties of γ can be seen in [16] and [4].

In this section $f\colon \bar{\Omega} \subset X \to X$ will be a mapping defined in the closure of a bounded and open subset Ω of X. It is always assumed that f is continuous and $f(\bar{\Omega})$ is bounded. The mapping f is an α-contraction if there exists a constant $k \in (0, 1)$ such that

$$\gamma(f(A)) \leqslant k\gamma(A)$$

for each subset A of $\bar{\Omega}$.

A typical example of α-contraction is $f = L + C$, where C is compact and L is a bounded linear operator with norm $|L| := \sup\{|Lx|/|x| \leqslant 1\} < 1$.

Since γ depends on the norm $|\cdot|$ the same will happen to the class of α-contractions. If we replace the norm $|\cdot|$ in X by another norm which is equivalent then the class of α-contractions will change. In [16] Nussbaum introduced a class of mappings which contains all mappings that are α-contractions with respect to some norm equivalent to $|\cdot|$. To define this class we need some notation.

Given a mapping $f: \bar{\Omega} \subset X \to X$, which is continuous and bounded, we define

$$K_1 = \overline{\mathrm{co}}\, f(\bar{\Omega}), \qquad K_n = \overline{\mathrm{co}}\, f(K_{n-1} \cap \bar{\Omega}), n > 1,$$

where $\overline{\mathrm{co}}\,(S)$ denotes the closed convex hull of the set S. The sets K_n are closed, convex and satisfy $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$. We define $K_\infty = \bigcap_{n=1}^{\infty} K_n$, which is also closed and convex. The mapping f is said to be like an α-contraction or to belong to the class $\mathfrak{L}C(\bar{\Omega})$ if K_∞ is compact. Sometimes the dependence of K_∞ with respect to f and Ω will be made explicit and we shall write $K_\infty(f, \Omega)$. An example of mapping in $\mathfrak{L}C(\bar{\Omega})$ is $f = L + C$, where C is compact and L is linear and has spectral radius $r(L) < 1$.

Given $f \in \mathfrak{L}C(\bar{\Omega})$, the set of fixed points $\mathrm{Fix}(f)$ is included in K_∞. The degree

$$\deg(I - f, \Omega; X)$$

is defined if f satisfies

$$\mathrm{Fix}(f) \cap \partial\Omega = \emptyset. \tag{2.1}$$

This degree is computed as the Leray-Schauder degree $\deg(I - f^*, \Omega; X)$, where $f^*: \bar{\Omega} \to X$ is any continuous mapping satisfying

$$f^*(x) = f(x) \text{ if } x \in K_\infty, \qquad f^*(\bar{\Omega}) \subset K_\infty.$$

More information on this degree can be seen in [16]. We present three properties that will be employed later.

Lemma 2.1 (Invariance by linear conjugation). *Let X and Y be Banach spaces and let $\Phi: X \to Y$ be a linear isomorphism. Assume that $f \in \mathfrak{L}C(\bar{\Omega})$ satisfies* (2.1) *and define*

$$g: \bar{\Omega} \subset Y \to Y, \qquad g = \Phi \circ f \circ \Phi^{-1}, \qquad \omega = \Phi(\Omega).$$

Then $g \in \mathfrak{L}C(\bar{\Omega})$, $Fix(g) \cap \partial\omega = \varnothing$ *and*

$$\deg(I - f, \Omega; X) = \deg(I - g, \omega; Y).$$

This is a consequence of the Commutativity Theorem in [16], Section D.

LEMMA 2.2 (Reduction of dimension). *Let* X_0 *be a closed subspace of* X *and let* $f \in \mathfrak{L}C(\bar{\Omega})$ *be a mapping satisfying* (2.1). *Assume also that* $f(\bar{\Omega}) \subset X_0$ *and define*

$$f_0 : \bar{\Omega} \cap X_0 \to X_0, \qquad x \mapsto f(x).$$

Then $f_0 \in \mathfrak{L}C(\bar{\Omega} \cap X_0)$ *and*

$$\deg(I - f, \Omega; X) = \deg(I - f_0, \Omega \cap X_0; X_0).$$

This is a very classical result when f is compact and deg refers to the Leray–Schauder degree. For α-contractions it is stated in [4]. The proof in our case is immediate if one uses the auxiliary mapping f^* which appeared in the definition of the degree of f.

The invariance of the degree with respect to homotopies in the class $\mathfrak{L}C(\bar{\Omega})$ is rather delicate (see Theorem 2 in Section D of [16]). Next we include a simpler result that will be sufficient for our purposes.

LEMMA 2.3. *Assume that* $f_1,\ f_2 \in \mathfrak{L}C(\bar{\Omega})$ *can be decomposed in the form*

$$f_i = L_i + C_i, \qquad i = 1, 2,$$

where L_i *is a bounded linear operator and* C_i *is compact. In addition,*

$$r(\lambda L_1 + (1 - \lambda) L_2) < 1 \tag{2.2}$$

and

$$\mathrm{Fix}(\lambda f_1 + (1 - \lambda) f_2) \cap \partial\Omega = \varnothing \tag{2.3}$$

for each $\lambda \in [0, 1]$. *Then*

$$\deg(I - f_1, \Omega; X) = \deg(I - f_2, \Omega; X).$$

Proof. The property (2.2) implies that for each $\lambda \in [0, 1]$ we can find a norm $|\cdot|_\lambda$ in X (equivalent to the norm $|\cdot|$) and a constant $K_\lambda \in [0, 1)$ such that

$$|(\lambda L_1 + (1 - \lambda) L_2)(x)|_\lambda \leqslant K_\lambda |x|_\lambda, \qquad x \in X.$$

By a compactness argument we can find a partition of $[0, 1]$, $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = 1$, and a constant $K \in [0, 1)$ such that if $\lambda \in [\lambda_i, \lambda_{i+1}]$ then

$$|(\lambda L_1 + (1-\lambda) L_2)(x)|_{\lambda_i} \leqslant K |x|_{\lambda_i}, \qquad x \in X.$$

The property of invariance by homotopies is well known in the class of α-contractions with respect to a fixed norm. Thus,

$$\deg(I - f_{\lambda_i}, \Omega; X) = \deg(I - f_{\lambda_{i+1}}, \Omega; X), \qquad i = 1, ..., n-1,$$

where $f_\lambda = \lambda f_1 + (1-\lambda) f_2$.

Remark. Given a bounded linear operator L with $r(L) < 1$ and a compact mapping C, the degree of $I - f$ with $f = L + C$ can also be defined using alternative techniques. For instance, one could transform the fixed point equation $x = f(x)$ into $x = f_*(x)$ with $f_* = (I - L)^{-1} \circ C$ and observe that f_* is compact. We shall need to vary L but when L is fixed this is related to the coincidence degree of Mawhin [6]. One of the authors (Ortega) thanks Massimo Furi for explaining to him the different ways of defining the degree of $I - f$.

3. THE LINEAR EQUATION

We use the general framework in [20]. Consider two Hilbert spaces H and V which are separable and such that

$$V \subset H$$

with compact inclusion. Moreover, V is dense in H (with respect to the H-topology). The norm and inner product in H will be simply denoted by $|\cdot|$ and $(\cdot, \cdot)$. In the case of V we will be more explicit and write $|\cdot|_V$ and $(\cdot, \cdot)_V$ respectively.

We shall also consider a bilinear form

$$a\colon V \times V \to \mathbb{R},$$

which is continuous, symmetric and coercive. In the usual way we associate to it an unbounded self-adjoint linear operator

$$\ell : \mathrm{dom}(\ell) \subset H \to H$$

with dom(ℓ) dense in H. The inverse of ℓ exists and is a compact operator from H into H. This allows us to apply the spectral theory of compact self-adjoint operators to ℓ^{-1} and also to construct the fractional powers ℓ^s in a simple way. More information on these powers can be seen in [20]. We just recall the identity $\mathrm{dom}(\ell^{1/2}) = V$.

Given $u_0 \in V$, $v_0 \in H$ and $p \in C([0, T]; H)$ we consider the initial value problem

$$\begin{cases} \ddot{u} + c\dot{u} + \ell u = p(t), & t \in (0, T), \\ u(0) = u_0, & \dot{u}(0) = v_0. \end{cases} \tag{3.1}$$

By a solution we understand a function $u \in \mathfrak{M} := C([0, T], V) \cap C^1([0, T], H)$ satisfying the initial conditions and such that for each $w \in V$ the equation below holds

$$\frac{d^2}{dt^2}(u(t), w) + c\frac{d}{dt}(u(t), w) + (\ell^{1/2}u(t), \ell^{1/2}w) = (p(t), w).$$

(This last expression is understood in the sense of distributions).

It is well known that (3.1) has a unique solution. It is also convenient to recall from [20], p. 180, the following fact: there exists $\varepsilon_0 > 0$ (depending only on c, a, V and H) such that if $0 < \varepsilon < \varepsilon_0$ then

$$\begin{aligned} |u(t)|_V^2 + |\dot{u}(t) + \varepsilon u(t)|^2 \leqslant \{ |u_0|_V^2 + |v_0 + \varepsilon u_0|^2 \} e^{-\varepsilon/2\, t} \\ + \frac{2}{\varepsilon^2}(1 - e^{-\varepsilon/2\, t}) \, |p|^2_{C([0, T]; H)}, \qquad t \in [0, T]. \end{aligned} \tag{3.2}$$

For small ε the formula

$$|(u_0, v_0)|_\varepsilon^2 := |u_0|_V^2 + |v_0 + \varepsilon u_0|^2, \qquad (u_0, v_0) \in V \times H,$$

defines a norm which is equivalent to any of the standard product norms in $V \times H$. The previous estimate (3.2) can be rephrased in terms of $|\cdot|_\varepsilon$.

The functional space $\mathfrak{M} = C([0, T], V) \cap C^1([0, T], H)$ is naturally endowed with the norm

$$\|u\|_{\mathfrak{M}} := \max_{t \in [0, T]} \{ |u(t)|_V + |\dot{u}(t)| \}.$$

Sometimes we shall use the equivalent norm

$$\|u\|_{\mathfrak{M}, \varepsilon} := \max_{t \in [0, T]} |(u(t), \dot{u}(t))|_\varepsilon,$$

for sufficiently small ε.

In this functional setting we define two linear operators associated to (3.1). Namely,

$$\mathscr{H}\colon V\times H\to \mathfrak{M}, \qquad (u_0, v_0)\mapsto u^*,$$

where u^* is the solution of (3.1) for $p=0$, and

$$\mathscr{M}\colon C([0,T];H)\to \mathfrak{M}, \qquad p\mapsto u^{**},$$

where u^{**} is the solution of (3.1) for $u_0=0$, $v_0=0$.

Notice that the solution of (3.1) for arbitrary (u_0, v_0) and p can be split as

$$u=u^*+u^{**}=\mathscr{H}(u_0,v_0)+\mathscr{M}p. \tag{3.3}$$

Also, notice that the inequality (3.2) implies that both operators $\mathscr{H}$ and $\mathscr{M}$ are continuous. At this point it is convenient to employ the norm $\|\cdot\|_{\mathfrak{M},\varepsilon}$.

In the rest of this Section we analyze the periodic problem

$$\begin{cases} \ddot{u}+c\dot{u}+\ell u=p(t), & t\in(0,T),\\ u(0)=u(T), & \dot{u}(0)=\dot{u}(T). \end{cases} \tag{3.4}$$

To this end we define the family of linear operators

$$\xi_\tau\colon \mathfrak{M}\to V\times H, \qquad u\mapsto (u(\tau),\dot{u}(\tau));\ \tau\in[0,T],$$

and the subspace of $\mathfrak{M}$ given by

$$\mathfrak{M}_T:=\{u\in\mathfrak{M}/\xi_0 u=\xi_T u\}.$$

The solutions of (3.4) are just functions in $\mathfrak{M}_T$ solving the differential equation in the sense already indicated.

LEMMA 3.1. *For each $p\in C([0,T];H)$ there exists a unique solution of (3.4). Moreover, the linear operator*

$$\mathscr{G}_T\colon p\in C([0,T];H)\mapsto u\in\mathfrak{M}_T$$

is continuous.

Proof. Given an arbitrary initial condition $(u_0,v_0)\in V\times H$, the solution of (3.1) is given by (3.3). From the identities

$$\xi_0\circ\mathscr{H}=\text{Identity}, \qquad \xi_0\circ\mathscr{M}=0, \tag{3.5}$$

we deduce that u will be periodic ($\xi_0 u = \xi_T u$) if and only if

$$(u_0, v_0) = \xi_T \mathscr{H}(u_0, v_0) + \xi_T \mathscr{M} p.$$

The operator $\xi_T \circ \mathscr{H}$ is a linear contraction with respect to the norm $|\cdot|_\varepsilon$. In fact the inequality (3.2) implies that

$$|\xi_T \mathscr{H}(u_0, v_0)|_\varepsilon \leqslant e^{-\varepsilon/4\,T}\, |(u_0, v_0)|_\varepsilon. \tag{3.6}$$

Thus the previous fixed point equation is uniquely solvable and (3.4) has a unique solution. To prove the continuity of $\mathscr{G}_T$ we just notice from (3.3) that it can be expressed in the form

$$\mathscr{G}_T = \mathscr{H} \circ (I - \xi_T \circ \mathscr{H})^{-1} \circ \xi_T \circ \mathscr{M} + \mathscr{M}. \tag{3.7}$$

We conclude this section with a concrete problem for which the abstract framework can be applied.

EXAMPLE 3.2. Define

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \qquad \mathbb{T}^n = \mathbb{T} \times \overset{(n}{\cdots} \times \mathbb{T},\ n \geqslant 1, \qquad H = L^2(\mathbb{T}^n), \qquad V = H^1(\mathbb{T}^n)$$

and

$$a(u, v) = \int_{\mathbb{T}^n} \nabla u \cdot \nabla v + \lambda u v,$$

where $\lambda > 0$ is fixed.

The associated operator ℓ is given by

$$\ell\colon \mathrm{dom}(\ell) \subset H \to H, \qquad \ell u = -\Delta u + \lambda u,$$

where $\mathrm{dom}(\ell) = H^2(\mathbb{T}^n)$.

The equation in (3.1) becomes the telegraph equation

$$u_{tt} + c u_t - \Delta u + \lambda u = p(t, x), \qquad t \in (0, T), \qquad x \in \mathbb{R}^n$$

with periodic boundary conditions

$$u(t, x_1, ..., x_i + 2\pi, ..., x_n) = u(t, x_1, ..., x_i, ..., x_n), \qquad 1 \leqslant i \leqslant n.$$

The function p belongs to $C([0, T], L^2(\mathbb{T}^n))$ and we know from Lemma 3.1 that this telegraph equation has a unique T-periodic solution.

4. THE NONLINEAR EQUATION

In this section we consider the differential equation

$$\ddot{u} + c\dot{u} + \ell u = f(t, u), \qquad t \in (0, T); \tag{4.1}$$

where ℓ is as in Section 3 and

$$f\colon [0, T] \times V \to H, \qquad (t, u) \mapsto f(t, u)$$

satisfies the following conditions:

(f-1) For each $r > 0$ there exists $\gamma_r > 0$ such that

$$|f(t, u_1) - f(t, u_2)| \leqslant \gamma_r |u_1 - u_2|_V$$

if $t \in [0, T]$ and $|u_1|_V, |u_2|_V \leqslant r$.

(f-2) For each $r > 0$ there exists a modulus of continuity $\omega_r > 0$ such that

$$|f(t_1, u_1) - f(t_2, u_2)| \leqslant \omega_r(|t_1 - t_2| + |u_1 - u_2|)$$

if $t_1, t_2 \in [0, T]$ and $|u_1|_V, |u_2|_V \leqslant r$.

(By a modulus of continuity we understand a function $\omega\colon [0, \infty) \to [0, \infty)$ which is increasing, continuous and such that $\omega(0) = 0$).

Remark. At first sight one could think that, when f does not depend on t, the modulus of continuity ω in (f-2) is linear. This seems to be suggested by (f-1). However this is not the case because the norm of $u_1 - u_2$ refers now to the space H.

Before we discuss the consequences of these assumptions we shall analyze them in a concrete case.

EXAMPLE 4.1. We continue with the notations of example 3.2 and consider now the semilinear telegraph equation

$$\begin{cases} u_{tt} + cu_t - \Delta_x u + \lambda u = F(t, x, u) \\ (t, x_1, ..., x_i + 2\pi, ..., x_n) = u(t, x_1, ..., x_i, ..., x_n), \ 1 \leqslant i \leqslant n \end{cases}$$

where $F\colon [0, T] \times \mathbb{T}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function. It will also be assumed that $\frac{\partial F}{\partial u}(t, x, u)$ exists and is continuous. Under certain additional conditions this equation can be interpreted as a particular case of (4.1) with

$$f(t, u) = F(t, \cdot, u(\cdot)), \qquad t \in [0, T], \qquad u \in H^1(\mathbb{T}^n).$$

When $n=1$ the space $H^1(\mathbb{T})$ is included in $C(\mathbb{T})$ and so f is always well defined and the conditions (f-1), (f-2) hold. When $n \geqslant 2$ we must impose additional conditions on F. We discuss the case $n \geqslant 3$. Assume

$$\left|\frac{\partial F}{\partial u}(t, x, u)\right| \leqslant k(1+|u|)^{\sigma-1},$$

$$|F(t_1, x, u) - F(t_2, x, u)| \leqslant (1+|u|)^{\sigma}\, \omega(|t_1 - t_2|)$$

where $k > 0$ is a constant, ω is a modulus of continuity and $\sigma \in (1, n/(n-2))$. To check that f is well defined we just recall Sobolev immersion $H^1(\mathbb{T}^n) \subset L^{n^*}(\mathbb{T}^n)$, $n^* = \frac{2n}{n-2}$. Next we notice that f is Hölder-continuous in u. This means that for each $r > 0$ there exists $\Gamma_r > 0$ such that

$$|f(t, u_1) - f(t, u_2)| \leqslant \Gamma_r |u_1 - u_2|^{\alpha}$$

if $t \in [0, T]$, $|u_1|_V$, $|u_2|_V \leqslant r$. Here $\alpha = 1 - \beta$ with $\beta = 2(\sigma - 1)/(n^* - 2)$. This property follows from Hölder inequality with $r = \frac{1}{\beta}$, $s = \frac{1}{\alpha}$, $\frac{1}{r} + \frac{1}{s} = 1$, because

$$|f(t, u_1) - f(t, u_2)|^2 \leqslant k^2 \int_{\mathbb{T}^n} (1 + |u_1| + |u_2|)^{2(\sigma-1)}\, |u_1 - u_2|^{2\beta}\, |u_1 - u_2|^{2\alpha}$$

$$\leqslant k^2 \left[\int_{\mathbb{T}^n} (1 + |u_1| + |u_2|)^{\frac{2(\sigma-1)}{\beta}}\, |u_1 - u_2|^2\right]^{\beta} \left[\int_{\mathbb{T}^n} |u_1 - u_2|^2\right]^{\alpha}.$$

The condition (f-1) is now easily checked. The condition (f-2) also follows from Hölder inequality and Sobolev immersions.

Let us go back to the general setting. A solution of (4.1) will be a function $u \in C^1(I, H) \cap C(I, V)$ satisfying, in the sense of Section 3, the linear equation

$$\ddot{u} + c\dot{u} + \ell u = p(t), \qquad t \in I$$

with $p(t) = f(t, u(t))$. Here I is some subinterval of $[0, T]$. The existence and uniqueness of local solution follows from a standard fixed point argument. Here one only uses (f-1) and the continuity of f. The following consequence of the proof will be employed several times.

Property L. Given $r > 0$ and $R > r$ there exists $\delta > 0$ such that if $u(t)$ is a solution of (4.1) with

$$|\dot{u}(t_0)| + |u(t_0)|_V \leqslant r,$$

for some t_0, then u can be continued up to $[t_0-\delta, t_0+\delta]\cap[0, T]$ and it satisfies on this interval

$$|\dot{u}(t)|+|u(t)|_V \leqslant R.$$

Next we shall obtain a result on continuous dependence with respect to weak topologies that requires (f-2). It is in the line of Proposition 5.1. in [1]. First we introduce some notation. Given $\tau\in(0, T]$ and $u\in C^1([0, \tau], H)\cap C([0, \tau], V)$,

$$|\|u\||_\tau=\max\{|\dot{u}(t)|+|u(t)|_V/t\in[0, \tau]\}.$$

Notice that $|\|\cdot\||_T$ coincides with $\|\cdot\|_{\mathfrak{M}}$.

Every solution $u(t)$ of (4.1) will be split in the form $u=u^{\#}+u^{\#\#}$ where $u^{\#}$ satisfies

$$\ddot{u}^{\#}+c\dot{u}^{\#}+\ell u^{\#}=0, \qquad u^{\#}(0)=u(0), \qquad \dot{u}^{\#}(0)=\dot{u}(0).$$

PROPOSITION 4.2. *Let $u_1, u_2, ..., u_n, ..., u_\infty$ be solutions of* (4.1) *defined in* $[0, \tau]$, $\tau\leqslant T$. *Assume that*

$$\sup_n |\|u_n\||_\tau<\infty$$

and

$$u_{0n}\rightharpoonup u_{0\infty} \text{ weakly in } V, \qquad v_{0n}\rightharpoonup v_{0\infty} \text{ weakly in } H,$$

where $u_n(0)=u_{0n}$, $\dot{u}_n(0)=v_{0n}$, $n=1, 2, ..., \infty$. *Then*

$$\max_{t\in[0,\tau]}|u_n(t)-u_\infty(t)|\to 0 \text{ and } |\|u_n^{\#\#}-u_\infty^{\#\#}\||_\tau\to 0 \qquad (n\to\infty).$$

Proof. First we notice that the sequence $\{u_n\}$ is relatively compact in $C([0, \tau], H)$. This is easily deduced from Ascoli Theorem, just as in the proof of Proposition 5.1. in [1]. It will be sufficient to prove the result for subsequences $\{u_k\}$ which are convergent in $C([0, \tau], H)$. The corresponding limit will be denoted by $u\in C([0, \tau], H)$. Define $p_k(t)=f(t, u_k(t))$. From (f-2) we deduce that $\{p_k\}$ is a Cauchy sequence in $C([0, \tau], H)$. Let $q(t)$ be its limit. If we adapt the notations of Section 3 to the interval $[0, \tau]$ then the decomposition $u_k=u_k^{\#}+u_k^{\#\#}$ can be expressed as

$$u_k^{\#}=\mathscr{H}(u_{0k}, v_{0k}), \qquad u_k^{\#\#}=\mathscr{M}p_k.$$

Linear equations behave well with respect to weak topologies and so we know that, for each $t\in[0, \tau]$,

$$u_k^{\#}(t)\rightharpoonup u_\infty^{\#}(t) \text{ weakly in } V, \qquad \dot{u}_k^{\#}(t)\rightharpoonup \dot{u}_\infty^{\#}(t) \text{ weakly in } H.$$

The continuity of $\mathscr{M}$ implies that $u = u_\infty^\# + \mathscr{M}q$ is in $C^1([0, T], H) \cap C([0, T], V)$ and $|\|u_k^{\#\#} - u^{\#\#}|\|_\tau \to 0$. It remains to prove that u and u_∞ coincide. This will follow by uniqueness because u is also a solution of (4.1). Actually, from (f-2),

$$|f(t, u_k(t)) - f(t, u(t))| \leqslant \omega_r(|u_k(t) - u(t)|), \qquad t \in [0, \tau],$$

for some $r > 0$. Therefore $q(t) = f(t, u(t))$ and so u is a solution of (4.1).

Given initial conditions (u_0, v_0) in $V \times H$, the solution of (4.1) satisfying $u(0) = u_0$, $\dot{u}(0) = v_0$ will be denoted by $u(t; u_0, v_0)$. This solution will be defined in a certain maximal interval and we define

$$\mathfrak{D} = \{(u_0, v_0) \in V \times H / u(t, u_0, v_0) \text{ is defined in } [0, T]\}.$$

The property L implies that $\mathfrak{D}$ is open. Define the solution operator

$$\Sigma\colon \mathfrak{D} \subset V \times H \to \mathfrak{M}, \qquad \Sigma(u_0, v_0) = u(\,\cdot\,; u_0, v_0).$$

It is clear that Σ is continuous. Perhaps more surprising is the fact that this operator is not necessarily bounded. This means that Σ can map a closed and bounded subset of $\mathfrak{D}$ onto an unbounded subset of $\mathfrak{M}$. Next example shows this.

Example 4.3. We start with the one-dimensional problem

$$\ddot{\xi} + c\dot{\xi} + \frac{c^2}{4}\,\xi = \lambda(t)\,\xi^3, \qquad \xi(0) = 1, \qquad \dot{\xi}(0) = 0. \tag{4.2}$$

It is possible to prove the existence of a number $\lambda^* > 0$ such that

- the solution of (4.2) is well defined in $[0, T]$ if $\lambda \in C([0, T])$ satisfies

$$0 \leqslant \lambda(t) \leqslant \lambda^*, \qquad t \in [0, T]$$

with strict inequality $\lambda(t) < \lambda^*$ somewhere,

- the solution of (4.2) for $\lambda(t) \equiv \lambda^*$ blows up at $t = T$.

(To prove these facts we found convenient to perform the change of variables $\xi = e^{-c/2t}\eta$. The system with unknowns η and $\dot{\eta}$ satisfies Kamke conditions and so the theory of differential inequalities applies).

Consider now the spaces $H = L^2(\mathbb{T})$, $V = H^1(\mathbb{T})$ and the operator $\ell u = -u_{xx} + \frac{c^2}{4}\,u$. Each function $u \in H$ is decomposed as $u = \bar{u} + \tilde{u}$ with $\bar{u} \in \mathbb{R}$ and $\int_{\mathbb{T}} \tilde{u} = 0$. This decomposition induces splittings $V = \mathbb{R} \oplus \tilde{V}$, $H = \mathbb{R} \oplus \tilde{H}$. Define

$$f(t, u) = \lambda^*(1 - \varepsilon\,|\tilde{u}|^2)\,\bar{u}^3,$$

where $\varepsilon > 0$ will be fixed later. The conditions (f-1) and (f-2) hold. Since f takes values onto the constant functions the solutions of (4.1) defined up to T can be decomposed as $u = \xi + \omega$ where $\omega \in C^1([0, T], \tilde{H}) \cap C([0, T], \tilde{V})$ satisfies

$$\ddot{\omega} + c\dot{\omega} + \ell\omega = 0$$

and $\xi: [0, T] \to \mathbb{R}$ is a solution of

$$\ddot{\xi} + c\dot{\xi} + \frac{c^2}{4}\xi = \lambda^* \left(1 - \varepsilon \int_{\mathbb{T}} \omega^2(t, x)\, dx\right) \xi^3.$$

Consider the set

$$C = \left\{(u_0, v_0) \in V \times H \Big/ \int_{\mathbb{T}} (u_0')^2 = 1, \bar{u}_0 = 1, v_0 = 0\right\}.$$

This set is closed and bounded and we can choose ε small enough so that any solution starting at C will satisfy $\varepsilon \int \omega^2 \leqslant 1$. Since ω is not identically zero if it has initial conditions on C, we deduce that $C \subset \mathfrak{D}$. Consider $(u_{0n}, 0) \in C$ with

$$u_{0n}(x) = 1 + \frac{\sin nx}{n\sqrt{\pi}}.$$

Then $\omega_n(t, x) = e^{-c/2\, t}(\cos nt + \frac{c}{2n} \sin nt)\, (\sin nx)/n\sqrt{\pi}$ and so $\int_{\mathbb{T}} \omega_n^2(t, x)\, dx \to 0$ as $n \to \infty$, uniformly in t. By continuous dependence we deduce that $\xi_n(t)$ converges to the solution of (4.2) for $\lambda \equiv \lambda^*$. Thus the sequence ξ_n cannot be bounded.

In the previous example the set C was not closed with respect to the weak topology. Next result shows that this fact was crucial.

PROPOSITION 4.4. *Let C be a bounded set in $V \times H$ which is closed with respect to the weak topology and, in addition, $C \subset \mathfrak{D}$. Then $\Sigma(C)$ is bounded in $\mathfrak{M}$.*

Proof. For each $\tau \in (0, T]$ define

$$\gamma(\tau) = \sup \{|\|u(\,\cdot\,; u_0, v_0)|\|_\tau; (u_0, v_0) \in C\}.$$

Property L implies that $\gamma(\tau)$ is finite for small τ. We want to prove that also $\gamma(T)$ is finite and we shall employ a contradiction argument. From now on we assume $\gamma(T) = \infty$. Define

$$T^* = \sup\{\tau \in (0, T]; \gamma(\tau) < \infty\}.$$

Again property L implies that $\gamma(T^*)=\infty$ and we can find a sequence (u_{0n}, v_{0n}) in C such that

$$|\|u_n\||_{T^*} \to \infty \qquad \text{where} \quad u_n = u(\,\cdot\,; u_{0n}, v_{0n}). \tag{4.3}$$

We can assume that the sequence (u_{0n}, v_{0n}) converges (in the weak sense) to a point $(u_{0\infty}, v_{0\infty})$ in C. This is possible because C is weakly closed. By assumption $C \subset \mathfrak{D}$ and so $u_\infty(t) = u(t; u_{0\infty}, v_{0\infty})$ can be defined in $[0, T]$. For each $\tau < T^*$ the norms $|\|u_n\||_\tau$ are bounded above and we can use Proposition 4.2 to deduce that $|u_n(t) - u_\infty(t)| \to 0$ uniformly in $t \in [0, \tau]$ and $|\|u_n^{\#\#} - u_\infty^{\#\#}\||_\tau \to 0$. The continuity of the operator $\mathscr{H}$ implies that $\|u_n^{\#}\|_{\mathfrak{M}}$ is bounded, say by a constant C. Let δ be the constant produced by property L for $r = C + 2\,\|u_\infty^{\#\#}\|_{\mathfrak{M}}$ and $R = 2r$. If we define $t_0 = T^* - \delta$ and assume that n is large enough then

$$|\dot{u}_n(t_0)| + |u_n(t_0)|_V \leqslant |\dot{u}_n^{\#}(t_0)| + |u_n^{\#}(t_0)|_V + |\dot{u}_n^{\#\#}(t_0)| + |u_n^{\#\#}(t_0)|_V \leqslant$$
$$C + 2\,\|u_\infty^{\#\#}\|_{\mathfrak{M}} = r.$$

Property L implies that

$$|\dot{u}_n(t)| + |u_n(t)|_V \leqslant R \qquad \text{if} \quad t \in (T^* - \delta, T^*].$$

In this way we arrive at the estimate

$$|\|u_n\||_{T^*} \leqslant \max\{\gamma(T^* - \delta), R\}$$

which is not compatible with (4.3).

5. TWO DEFINITIONS OF INDEX

Throughout the rest of the paper we always assume that ℓ is in the conditions of Section 3 and f satisfies (f-1) and (f-2). We shall be interested in the periodic solutions of the equation (4.1). These are the solutions lying in $\mathfrak{M}_T$ or, equivalently, satisfying the periodic boundary conditions

$$u(0) = u(T), \qquad \dot{u}(0) = \dot{u}(T). \tag{5.1}$$

5.1. *The Poincaré Map*

It is defined by

$$\mathscr{P} : \mathfrak{D} \subset V \times H \to V \times H, \qquad \mathscr{P}(u_0, v_0) = (u(T; u_0, v_0), \dot{u}(T; u_0, v_0))$$

or, equivalently,

$$\mathscr{P} = \xi_T \circ \Sigma \tag{5.2}$$

The fixed points of $\mathscr{P}$ are the initial conditions of the periodic solutions of (4.1).

We know from the previous section that $\mathscr{P}$ is continuous but not necessarily bounded. Let G be an open and bounded subset of $V \times H$. The closure of G with respect to the weak topology will be denoted by $cl_w(G)$. We recall that the strong closure $\bar{G}$ is always included in $cl_w(G)$ and the identity $\bar{G} = cl_w(G)$ holds at least when G is convex. If $cl_w(G) \subset \mathfrak{D}$ we can apply Proposition 4.4 with $C = cl_w(G)$ to conclude that $\mathscr{P}(\bar{G})$ is bounded.

Proposition 5.1. *Let G be an open and bounded subset of $V \times H$ such that*

$$cl_w(G) \subset \mathfrak{D}.$$

Then the map $\mathscr{P}$ is in the class $\mathfrak{L}C(\bar{G})$.

Proof. We know that $\Sigma(\bar{G})$ is a bounded subset of $\mathfrak{M}$. Going back to the notations of Section 3 we observe that $u(t; u_0, v_0)$ solves the linear problem (3.1) with $p = \mathscr{N}\Sigma(u_0, v_0)$ where $\mathscr{N}$ is the substitution operator

$$\mathscr{N}: \mathfrak{M} \to C([0, T], H), \qquad u \mapsto f(\cdot, u(\cdot)).$$

From (3.3) we deduce the identity

$$\Sigma = \mathscr{H} + \mathscr{M} \circ \mathscr{N} \circ \Sigma. \tag{5.3}$$

From now on this identity will be referred as the abstract Volterra equation. Using (5.2) we rewrite $\mathscr{P}$ as

$$\mathscr{P} = \xi_T \circ \mathscr{H} + \xi_T \circ \mathscr{M} \circ \mathscr{N} \circ \Sigma.$$

We already found in the proof of Lemma 3.1 that $\xi_T \circ \mathscr{H}$ is a linear contraction with respect to some norm $|\cdot|_\varepsilon$. The proof of this proposition will be completed if we show that $\mathscr{M} \circ \mathscr{N} \circ \Sigma$ is compact on $\bar{G}$. This is a consequence of the lemma below.

Lemma 5.2. *The operator $\mathscr{N}$ is completely continuous.*

Proof. We know from (f-1) and (f-2) that $\mathscr{N}$ is continuous and maps bounded sets into bounded sets. Let $\mathfrak{B}$ be a bounded set of $\mathfrak{M}$. We must proof that $\mathscr{N}(\mathfrak{B})$ is relatively compact in $C([0, T], H)$. To this end we apply Ascoli Theorem and so we must verify the two conditions below.

(i) $\mathcal{N}(\mathfrak{B})$ is equicontinuous;

(ii) $\{f(t, u(t))/u \in \mathfrak{B}\}$ is relatively compact in H for each $t \in [0, T]$.

Let $r > 0$ be such that $\max_{t \in [0, T]} |u(t)|_V \leqslant r$ for each $u \in \mathfrak{B}$. To prove (i) we notice that if $t_1, t_2 \in [0, T]$ then

$$|u(t_1) - u(t_2)| \leqslant d\, |t_1 - t_2|, \qquad u \in \mathfrak{B},$$

with $d = \max\{|\dot{u}(t)|/t \in [0, T], u \in \mathfrak{B}\}$. From (f-2) we deduce that

$$|\mathcal{N} u(t_1) - \mathcal{N} u(t_2)| \leqslant \omega_r((1+d)\, |t_1 - t_2|).$$

To prove (ii) we consider the closed ball of radius r in V,

$$B_r = \{u \in V/|u|_V \leqslant r\}.$$

We look at B_r as a metric space immersed in H (with the topology induced by $|\cdot|$). An argument of weak compactness shows that B_r is compact and the condition (f-2) says that f: $[0, T] \times B_r \to H$ is continuous. Thus $f(t, B_r)$ is compact and (ii) follows.

We are now in a position to associate a degree to $I - \mathscr{P}$. Let G be a bounded and open subset of $V \times H$ satisfying

$$cl_w(G) \subset \mathfrak{D} \qquad \text{and} \qquad \operatorname{Fix}(\mathscr{P}) \cap \partial G = \varnothing. \tag{5.4}$$

Then we can define the degree

$$\deg(I - \mathscr{P}, G;\, V \times H).$$

Given $u \in \mathfrak{M}_T$ periodic solution of (4.1), we say that it is isolated if there exists a neighborhood N of $(u(0), \dot{u}(0))$ in $V \times H$ such that $\operatorname{Fix}(\mathscr{P}) \cap \bar{N} = \{(u(0), \dot{u}(0))\}$. The index of u is defined as

$$\gamma_{\mathscr{P}}(u) := \deg\, (I - \mathscr{P}, N;\, V \times H).$$

Notice that we can always choose N small enough so that $cl_w(N) \subset \mathfrak{D}$. Moreover, the properties of the degree imply that this definition is independent of the choice of N.

5.2. *The Functional-Analytic Approach*

The restriction of $\mathcal{N}$ to $\mathfrak{M}_T$ will be indicated by $\mathcal{N}_T$. Define the operator

$$\mathscr{F}_T : \mathfrak{M}_T \to \mathfrak{M}_T, \qquad \mathscr{F}_T = \mathscr{G}_T \circ \mathcal{N}_T.$$

The definition of $\mathscr{G}_T$ implies that the fixed points of $\mathscr{F}_T$ are precisely the solutions of (4.1) lying in $\mathfrak{M}_T$. Moreover, Lemma 5.2 implies that $\mathscr{F}_T$ is completely continuous and so we can define the degree

$$\deg(I-\mathscr{F}_T, \Omega_T; \mathfrak{M}_T)$$

when Ω_T is an open and bounded subset of $\mathfrak{M}_T$ such that

$$\partial\Omega_T \cap \mathrm{Fix}(\mathscr{F}_T) = \emptyset. \tag{5.5}$$

A solution $u \in \mathfrak{M}_T$ is isolated (in the sense previously defined) if and only if there exists a neighborhood ω_T of u in $\mathfrak{M}_T$ such that $\mathrm{Fix}(\mathscr{F}_T) \cap \overline{\omega_T} = \{u\}$. This is a consequence of the continuity of Σ. Thus, for an isolated solution u in $\mathfrak{M}_T$ we can define the second index

$$\gamma_{\mathscr{F}}(u) := \deg(I-\mathscr{F}_T, \omega_T; \mathfrak{M}_T).$$

In principle this definition depends on the choice of ℓ. Given $\lambda \in \mathbb{R}$ such that the quadratic form $a(u, u) + \lambda|u|^2$ is coercive, we can rewrite (4.1) in the form

$$\ddot{u} + c\dot{u} + \ell_\lambda u = f_\lambda(t, u), \tag{5.6}$$

where $\ell_\lambda = \ell + \lambda I$ and $f_\lambda(t, u) = f(t, u) + \lambda u$. The operator ℓ_λ and the function f_λ satisfy all the previous requirements and so we can define the index in terms of (5.6). However, (5.6) clearly define a homotopy in Ω_T or ω_T and so the definitions of degree and index are independent of λ. Sometimes the index is defined in terms of (5.6) for a value of λ which is in the resolvent of $-\ell$ but such that $a(\cdot, \cdot) + \lambda\,|\cdot|$ is not coercive. In these cases the index can have a reversed sign (see [17]).

6. THE DUALITY THEOREM

Let $G \subset V \times H$ and $\Omega_T \subset \mathfrak{M}_T$ be two bounded and open sets. We assume that the weak closure of G, $cl_w(G)$, is inside $\mathfrak{D}$ (the domain of $\mathscr{P}$). Following [11] we say that G and Ω_T have a *common core* with respect to the periodic problem (4.1)–(5.1) if the conditions below hold,

$$\mathrm{Fix}(\mathscr{F}_T) \cap \partial\Omega_T = \emptyset, \qquad \mathrm{Fix}(\mathscr{P}) \cap \partial G = \emptyset, \tag{6.1}$$

$$\xi_0(\mathrm{Fix}(\mathscr{F}_T) \cap \Omega_T) = \mathrm{Fix}(\mathscr{P}) \cap G, \tag{6.2}$$

$$\Sigma(\mathrm{Fix}(\mathscr{P}) \cap G) = \mathrm{Fix}(\mathscr{F}_T) \cap \Omega_T. \tag{6.3}$$

The condition (6.1) says that there are no T-periodic solutions lying on $\partial\Omega_T$ or having initial conditions on ∂G. The conditions (6.2), (6.3) are equivalent to saying that a periodic solution belongs to Ω_T if and only if its initial condition belongs to G.

THEOREM 6.1. *Assume that G and Ω_T are in the previous conditions and have a common core with respect to* (4.1)–(5.1). *Then*

$$\deg(I-\mathscr{P}, G; V\times H) = \deg(I-\mathscr{F}_T, \Omega_T; \mathfrak{M}_T).$$

In particular, given an isolated T-periodic solution u,

$$\gamma_{\mathscr{P}}(u) = \gamma_{\mathscr{F}}(u).$$

Before presenting the proof of this result we discuss some consequences. More specifically we discuss some connections between the index and the stability properties of a periodic solution. In principle the equation (4.1) is only defined in the time interval $[0, T]$ and the classical notions in stability theory do not make sense. However we shall assume now that $f(0, \cdot)$ and $f(T, \cdot)$ coincide and so f can be extended by periodicity. We say that a T-periodic solution $u(t)$ is stable (in the Lyapunov sense) if given any neighborhood $\mathscr{U}$ of $(u(0), \dot{u}(0))$ in $V\times H$ it is possible to find another neighborhood $\mathscr{M}$ such that if $(u_0, v_0)\in\mathscr{M}$ then $u(t; u_0, v_0)$ is well defined for $t\in[0, \infty)$ and $(u(t; u_0, v_0), \dot{u}(t; u_0, v_0))\in\mathscr{U}$ for each $t\geqslant 0$. The solution $u(t)$ is asymptotically stable if it is stable and there is a neighborhood $\mathscr{W}$ of $(u(0), \dot{u}(0))$ such that

$$|u(t; u_0, v_0) - u(t)|_V + |\dot{u}(t; u_0, v_0) - \dot{u}(t)| \to 0 \qquad \text{as} \quad t\to +\infty$$

for each $(u_0, v_0)\in\mathscr{W}$.

From the definition we can see that an asymptotically stable periodic solution is isolated and we want to compute its index. To this end we notice that $u(t)$ is asymptotically stable if and only if the initial condition $(u(0), \dot{u}(0))$ has the same property as a fixed point of the Poincaré map $\mathscr{P}$. This map is well defined in some neighborhood of $(u(0), \dot{u}(0))$ and we know from the proof of the Proposition 5.1 that $\mathscr{P}$ is an α-contraction with respect to the norm $|\cdot|_\varepsilon$. Thus the asymptotic stability is uniform (see [7]). In consequence we can find an open and bounded neighborhood N of $(u(0), \dot{u}(0))$ satisfying the following properties

$$\mathscr{P}(N)\subseteq N, \qquad \operatorname{Fix}(\mathscr{P})\cap\bar{N} = \bigcap_{n\geqslant 0}\mathscr{P}^n(\bar{N}) = \{(u(0), \dot{u}(0))\}.$$

We can now apply the asymptotic fixed point theorem in [15] to deduce that

$$\deg\,(I-\mathscr{P}, N;\, V\times H)=1.$$

This leads to the following result.

COROLLARY 6.2. *Let $u(t)$ be an asymptotically stable T-periodic solution of* (4.1). *Then*

$$\gamma_{\mathscr{P}}(u)=\gamma_{\mathscr{F}}(u)=1.$$

In the case of ordinary differential equations this result follows from Theorem 9.6. in [[10], Chap. III] and the Duality Theorem of [11].

The rest of the Section is devoted to the proof of Theorem 6.1

An outline of the proof. We follow along the lines of Theorem 28.5 in [11]. The first difficulty to connect the degrees of $I-\mathscr{F}_T$ and $I-\mathscr{P}$ is that these operators are defined in different spaces. This is overcame by constructing new operators $\mathscr{F}$ and $\mathscr{P}_\sharp$ which are defined on the common space $\mathfrak{M}$ and preserve the corresponding degrees. The operator $\mathscr{F}$ will be a straightforward extension of $\mathscr{F}_T$. To construct $\mathscr{P}_\sharp$ we shall immerse $V\times H$ into $\mathfrak{M}$ and then transport $\mathscr{P}$ to $\mathfrak{M}$ by conjugation. Once $\mathscr{F}$ and $\mathscr{P}_\sharp$ have been defined we define two auxiliary operators $\mathscr{A}_1$ and $\mathscr{A}_2$ and find a homotopy in three steps, according to the figure below:

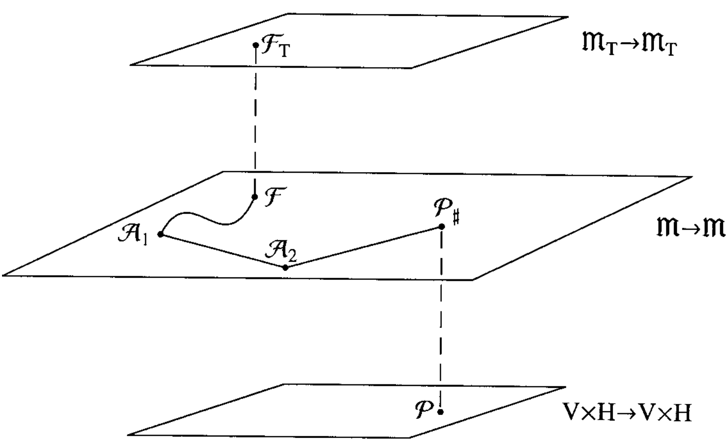

FIGURE 1

Throughout the proof we shall assume that the following additional condition holds,

$$\operatorname{Fix}(\mathscr{F}_T) \cap \Omega_T \neq \varnothing \tag{6.4}$$

Notice that if it does not hold then the Theorem is trivially satisfied with

$$\deg(I - \mathscr{P}, G; V \times H) = \deg(I - \mathscr{F}_T, \Omega_T; \mathfrak{M}_T) = 0.$$

6.1. *The Operator* $\mathscr{F}$

Define

$$\mathscr{F} : \mathfrak{M} \to \mathfrak{M}, \qquad \mathscr{F}(u) = \mathscr{G}_T \circ \mathscr{N}(u).$$

As in the case of $\mathscr{F}_T$ one proves that $\mathscr{F}$ is completely continuous using Lemmas 3.1 and 5.2 Moreover,

$$\mathscr{F}(\mathfrak{M}) \subseteq \mathfrak{M}_T \text{ and } \mathscr{F}(u) = \mathscr{F}_T(u) \qquad \forall u \in \mathfrak{M}_T.$$

This implies in particular,

$$\operatorname{Fix}(\mathscr{F}) = \operatorname{Fix}(\mathscr{F}_T).$$

LEMMA 6.3. *There exists a nonempty subset* ω *of* $\mathfrak{M}$, *which is open and bounded, satisfying the conditions*:

$$\operatorname{Fix}(\mathscr{F}) \cap \omega = \operatorname{Fix}(\mathscr{F}_T) \cap \Omega_T, \tag{6.5}$$

$$\operatorname{Fix}(\mathscr{F}) \cap \partial\omega = \varnothing, \tag{6.6}$$

$$cl_w \xi_0(\omega) \subset \mathfrak{D}, \tag{6.7}$$

where $\mathfrak{D}$ *is the domain of the Poincaré operator* $\mathscr{P}$.

Proof. The set $\mathscr{K} := \operatorname{Fix}(\mathscr{F}_T) \cap \overline{\Omega_T}$ is compact and nonempty. Here we have used (6.4). Moreover $\xi_0(\mathscr{K})$ is contained in the open set $\mathfrak{D}$. Thus we can find a finite number of balls in $\mathfrak{M}$, $B_1, ..., B_r$, covering $\mathscr{K}$ and such that $cl_w \xi_0(B_i) \subset \mathfrak{D}$, $1 \leqslant i \leqslant r$. Moreover we can choose these balls small enough so that

$$\omega := \bigcup_{i=1}^{r} B_i$$

satisfies (6.5), (6.6) and (6.7).

Define $\omega_T = \omega \cap \mathfrak{M}_T$. The properties (6.5) and (6.6) imply that

$$\deg(I - \mathscr{F}_T, \Omega_T; \mathfrak{M}_T) = \deg(I - \mathscr{F}_T, \omega_T; \mathfrak{M}_T). \tag{6.8}$$

In view of Lemma 2.2 we can say that this last degree coincides with

$$\deg(I - \mathscr{F}, \omega; \mathfrak{M}).$$

6.2. *The Operator* $\mathscr{P}_\sharp$

First we are going to construct an immersion of $V \times H$ into $\mathfrak{M}$. To this end we consider a fixed smooth function (C^1 is sufficient) φ: $[0, T] \to \mathbb{R}$ satisfying the boundary conditions

$$\varphi(0) = 1, \qquad \varphi(T) = \dot{\varphi}(0) = \dot{\varphi}(T) = 0.$$

Also we introduce the linear operators

$$\mathscr{H}_\tau: V \times H \to \mathfrak{M}, \qquad u = \mathscr{H}_\tau(u_0, v_0),$$

where $\tau \in [0, T]$ and u is the solution of the initial value problem

$$\ddot{u} + c\dot{u} + \ell u = 0, \qquad u(\tau) = u_0, \quad \dot{u}(\tau) = v_0.$$

We notice that $\mathscr{H}_0$ is precisely the operator $\mathscr{H}$ introduced in Section 3. A variant of the inequality (3.2) proves that $\mathscr{H}_\tau$ is continuous.

We can define

$$j: V \times H \to \mathfrak{M}, \qquad j(u_0, v_0) = \varphi \mathscr{H}_0(u_0, v_0) + (1 - \varphi)\, \mathscr{H}_T(u_0, v_0).$$

This is a bounded linear operator satisfying

$$\xi_0 \circ j = \xi_T \circ j = \text{Identity in } V \times H. \tag{6.9}$$

The image $\mathfrak{M}_* := j(V \times H)$ is contained in $\mathfrak{M}_T$ (i.e., $\xi_{0|\mathfrak{M}_*} = \xi_{T|\mathfrak{M}_*}$) and it is clear that j induces an isomorphism between $V \times H$ and $\mathfrak{M}_*$. In particular,

$$j \circ \xi_{0|\mathfrak{M}_*} = I_{\mathfrak{M}_*}.$$

Also we notice that $\mathfrak{M}_*$ is closed in $\mathfrak{M}$ (or in $\mathfrak{M}_T$).

We can now transport $\mathscr{P}$ to $\mathfrak{M}_*$. Define

$$\mathfrak{D}_* = \{u \in \mathfrak{M}_* / \xi_0 u \in \mathfrak{D}\}$$

and

$$\mathscr{P}_*: \mathfrak{D}_* \subset \mathfrak{M}_* \to \mathfrak{M}_*, \qquad \mathscr{P}_*(u) = j \circ \mathscr{P} \circ \xi_0(u).$$

Define $G_* = j(G)$, it follows from Lemma 2.2 and Proposition 5.1 that $\mathscr{P}_* \in \mathfrak{L}C(\overline{G_*})$ and

$$\deg(I - \mathscr{P}, G; V \times H) = \deg(I - \mathscr{P}_*, G_*; \mathfrak{M}_*). \tag{6.10}$$

Given $r > 0$ we are going to consider the open and bounded subset of $\mathfrak{M}$,

$$G_\sharp(r) = \{u \in \mathfrak{M} / \xi_0 u \in G,\ \|u\|_{\mathfrak{M}} < r\}.$$

We can choose r large enough so that

$$G_* = G_\sharp(r) \cap \mathfrak{M}_*, \qquad \partial G_* = \partial G_\sharp(r) \cap \mathfrak{M}_*, \tag{6.11}$$

where ∂G_* is the boundary of G_* in $\mathfrak{M}_*$.

Define $\mathfrak{D}_\sharp = \{u \in \mathfrak{M} / \xi_0 u \in \mathfrak{D}\}$ and

$$\mathscr{P}_\sharp: \mathfrak{D}_\sharp \subset \mathfrak{M} \to \mathfrak{M}, \qquad \mathscr{P}_\sharp = j \circ \mathscr{P} \circ \xi_0.$$

This map belongs to $\mathfrak{L}C(\overline{G_\sharp(r)})$ because

$$K_\infty(\mathscr{P}_\sharp, G_\sharp(r)) = K_\infty(\mathscr{P}_*, G_*).$$

Moreover $\mathscr{P}_\sharp$ is an extension of $\mathscr{P}_*$ with $\mathscr{P}_\sharp(\mathfrak{D}_\sharp) \subset \mathfrak{M}_*$. From (6.11) we deduce that $\mathscr{P}_\sharp$ has no fixed points on the boundary of $G_\sharp(r)$ and so, from Lemma 2.2,

$$\deg(I - \mathscr{P}_\sharp, G_\sharp(r); \mathfrak{M}) = \deg(I - \mathscr{P}_*, G_*; \mathfrak{M}_*). \tag{6.12}$$

6.3. *The auxiliary operators*

Define

$$\mathscr{A}_1: \mathfrak{M} \to \mathfrak{M}, \qquad \mathscr{A}_1 = \mathscr{H}_0 \circ \xi_T + \mathscr{M} \circ \mathscr{N}$$

and

$$\mathscr{A}_2: \mathfrak{D}_\sharp \subset \mathfrak{M} \to \mathfrak{M}, \qquad \mathscr{A}_2 = \mathscr{H}_0 \circ \xi_T + \mathscr{M} \circ \mathscr{N} \circ \Sigma \circ \xi_0.$$

Let ω be the domain given by Lemma 6.3 The condition (6.7) implies that $\mathscr{A}_2$ is well defined on $\bar{\omega}$ and $\mathscr{A}_2(\bar{\omega})$ is a bounded set. Clearly the operator $\mathscr{A}_1$ is well defined on the whole space $\mathfrak{M}$. Our first task will be to prove that $\mathscr{A}_1$ and $\mathscr{A}_2$ belong to $\mathfrak{L}C(\bar{\omega})$. The operators $\mathscr{M} \circ \mathscr{N}$ and $\mathscr{M} \circ \mathscr{N} \circ \Sigma \circ \xi_0$ are compact on $\bar{\omega}$ and therefore it will be sufficient to apply the following result.

LEMMA 6.4. *The spectral radius of* $\mathcal{H}_0 \circ \xi_T \colon \mathfrak{M} \to \mathfrak{M}$ *satisfies*

$$r(\mathcal{H}_0 \circ \xi_T) < 1.$$

Proof. First we consider the operator $\xi_T \circ \mathcal{H}_0$ and notice that the estimate (3.6) implies that the spectral radius of this operator satisfies

$$r(\xi_T \circ \mathcal{H}_0) < 1.$$

This is equivalent to saying that the spectrum of $\xi_T \mathcal{H}_0$ lies inside the unit disk.

Given a number $\lambda \in \mathbb{C} - \{0\}$ in the resolvent set of $\xi_T \mathcal{H}_0$, it is easy to prove that λ is also in the resolvent of $\mathcal{H}_0 \xi_T$ and the following formula holds

$$(\mathcal{H}_0 \xi_T - \lambda I)^{-1} = \frac{1}{\lambda} [\mathcal{H}_0 (\xi_T \mathcal{H}_0 - \lambda I)^{-1} \xi_T - I].$$

Thus the spectrum of $\mathcal{H}_0 \xi_T$ also lies inside the unit disk and the proof is complete.

Next result implies in particular that $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same fixed points as $\mathcal{F}$.

LEMMA 6.5. *For each* $\lambda \in [0, 1]$

$$\mathrm{Fix}(\lambda \mathcal{A}_1 + (1-\lambda)\, \mathcal{A}_2) = \mathrm{Fix}(\mathcal{F}).$$

Proof. Let u be a fixed point of $\lambda \mathcal{A}_1 + (1-\lambda)\, \mathcal{A}_2$. If we define $\hat{u} := \Sigma \xi_0 u$ then $\hat{u}$ is a solution of (4.1) satisfying the same initial conditions as u ($\xi_0 \hat{u} = \xi_0 u$). Let us consider the equation

$$\ddot{w} + c\dot{w} + \ell w = g(t, w), \tag{6.13}$$

where $g(t, w) := (1-\lambda)\, f(t, w) + \lambda f(t, \hat{u}(t))$. The function g satisfies the conditions (f-1) and (f-2) and so there is uniqueness for the initial value problem associated to (6.13). The definition of g implies that $\hat{u}$ is a solution of (6.13). On the other hand we know that u is a fixed point of $\lambda \mathcal{A}_1 + (1-\lambda)\, \mathcal{A}_2$ and so it satisfies

$$u = \mathcal{H}_0 \xi_T u + \lambda \mathcal{M} \mathcal{N} u + (1-\lambda)\, \mathcal{M} \mathcal{N} \Sigma \xi_0 u. \tag{6.14}$$

From this equation and (3.5) we deduce that u is periodic; that is, $\xi_0 u = \xi_T u$. Now (6.14) can be rewritten as

$$u = \mathcal{H}_0 \xi_0 u + \mathcal{M} g(\cdot, u).$$

From (3.3) we deduce that u is a solution of (6.13). By uniqueness we deduce that $u = \hat{u}$ and so u is a periodic solution of (4.1). This implies $\mathrm{Fix}(\lambda \mathscr{A}_1 + (1-\lambda)\, \mathscr{A}_2) \subset \mathrm{Fix}(\mathscr{F})$. To prove the other inclusion we notice that $u \in \mathrm{Fix}(\mathscr{F})$ implies the identities $u = \Sigma \xi_0 u$, $\xi_0 u = \xi_T u$ and $u = \mathscr{H}_0 \xi_T u + \lambda \mathscr{M} \mathscr{N} u + (1-\lambda)\, \mathscr{M} \mathscr{N} \Sigma \xi_0 u$.

Once we have proved this lemma we can use (6.6) to deduce that

$$\deg(I - \lambda \mathscr{A}_1 - (1-\lambda)\, \mathscr{A}_2, \omega; \mathfrak{M}) \tag{6.15}$$

is well defined and constant. Here we have used Lemma 2.3

6.4. *Connecting $\mathscr{F}$ to $\mathscr{A}_1$*

Since $\mathscr{F}(\mathfrak{M}) \subset \mathfrak{M}_T$ we know that $\xi_T \mathscr{F} = \xi_0 \mathscr{F}$. Thus,

$$(I - \mathscr{H}_0 \xi_T)(I - \mathscr{F}) = I - \mathscr{H}_0 \xi_T - \mathscr{F} + \mathscr{H}_0 \xi_0 \mathscr{F}.$$

From the definition of $\mathscr{F}$ and (3.7)

$$\mathscr{F} = [\mathscr{H}_0 (I - \xi_T \mathscr{H}_0)^{-1} \xi_T + I]\, \mathscr{M} \mathscr{N}.$$

Combining this identity with (3.5),

$$\mathscr{H}_0 \xi_0 \mathscr{F} = \mathscr{H}_0 (I - \xi_T \mathscr{H}_0)^{-1} \xi_T \mathscr{M} \mathscr{N}.$$

These three identities lead us to

$$(I - \mathscr{H}_0 \xi_T)(I - \mathscr{F}) = I - \mathscr{A}_1.$$

Then

$$H_\lambda := (I - \lambda \mathscr{H}_0 \xi_T)(I - \mathscr{F}) = I - \lambda \mathscr{H}_0 \xi_T - \mathscr{F} + \lambda \mathscr{H}_0 \xi_T \mathscr{F}, \qquad \lambda \in [0, 1],$$

defines a homotopy between $I - \mathscr{F}$ and $I - \mathscr{A}_1$. For each $\lambda \in [0, 1]$, $I - H_\lambda$ is in $\mathfrak{L}C(\bar{\omega})$ and $\mathrm{Fix}(I - H_\lambda)$ is independent of λ. All these properties and Lemma 2.3 lead to the conclusion

$$\deg(I - \mathscr{F}, \omega; \mathfrak{M}) = \deg(I - \mathscr{A}_1, \omega; \mathfrak{M}). \tag{6.16}$$

6.5. *Connecting $\mathscr{A}_2$ to $\mathscr{P}_\#$*

We intend to apply again Lemma 2.3. To this end we notice that $\mathscr{A}_2$ can be decomposed as

$$\mathscr{A}_2 = L_1 + \mathscr{C}_1, \qquad L_1 = \mathscr{H}_0 \circ \xi_T, \qquad \mathscr{C}_1 = \mathscr{M} \circ \mathscr{N} \circ \Sigma \circ \xi_0,$$

and $\mathscr{C}_1$ is compact on $\bar{\omega}$.

Also, from the definition of $\mathscr{P}_\#$ and the identities (5.2) and (5.3), one obtains

$$\mathscr{P}_\# = j\mathscr{P}\xi_0 = j\xi_T \Sigma \xi_0 = j\xi_T(\mathscr{H}_0 + \mathscr{M}\mathscr{N}\Sigma)\,\xi_0 = L_2 + \mathscr{C}_2,$$

where

$$L_2 = j \circ \xi_T \circ \mathscr{H}_0 \circ \xi_0 \qquad \text{and} \qquad \mathscr{C}_2 = j \circ \xi_T \circ \mathscr{M} \circ \mathscr{N} \circ \Sigma \circ \xi_0.$$

The operator $\mathscr{C}_2$ is compact on $\bar{\omega}$ and we shall verify (2.2) and (2.3) by means of three lemmas. The first of them is an easy exercise in Functional Analysis.

LEMMA 6.6. *Let X be a Banach space with norm $|\cdot|$ and let $L: X \to X$ be a bounded linear operator. Assume that there exists a semi-norm $\|\cdot\|$ in X such that the conditions below hold*

$$\text{(i)} \quad k\,|x| \leqslant \|x\| \leqslant K\,|x|, \qquad \forall x \in \operatorname{Im}(L),$$

$$\text{(ii)} \quad \|Lx\| \leqslant \Gamma\,\|x\|, \qquad \forall x \in X,$$

where k, K and Γ are fixed positive constants.

Then the spectral radius of L satisfies

$$r(L) \leqslant \Gamma.$$

Proof. We prove the inequality

$$|L^n| \leqslant \frac{K}{k}\,\Gamma^{n-1}\,|L|, \qquad n = 1, 2, ...$$

and the result is a consequence of Gelfand's formula for the spectral radius. Given an arbitrary $y \in X$ we observe that (i) holds for $x = Ly$ or $L^n y$, $n \geqslant 2$. Thus

$$|L^n y| \leqslant \frac{1}{k}\,\|L^n y\| \leqslant \frac{\Gamma^{n-1}}{k}\,\|Ly\| \leqslant \frac{K\Gamma^{n-1}}{k}\,|Ly| \leqslant \frac{K\Gamma^{n-1}}{k}\,|L|\,|y|.$$

LEMMA 6.7. *For each $\lambda \in [0, 1]$,*

$$r(L_\lambda) < 1,$$

where $L_\lambda = \lambda \mathscr{H}_0 \circ \xi_T + (1-\lambda)\, j \circ \xi_T \circ \mathscr{H}_0 \circ \xi_0$.

Proof. We already know from Lemma 6.4 that the result is valid for $\lambda = 1$. When $\lambda = 0$ we notice that the powers of L_0 satisfy

$$L_0^n = j(\xi_T \mathscr{H}_0)^n \xi_0.$$

Since $r(\xi_T \mathscr{H}_0) < 1$ we can conclude that the result also holds for $\lambda = 0$. From now on we assume $\lambda \in (0, 1)$.

We are going to apply the previous Lemma with $X = \mathfrak{M}$, $|\cdot| = \|\cdot\|_{\mathfrak{M}, \varepsilon}$ and the semi-norm $\|\cdot\|$ defined by

$$\|u\|_\partial := \max\{|\xi_0 u|_\varepsilon, |\xi_T u|_\varepsilon\}, \qquad u \in \mathfrak{M}.$$

Here $\varepsilon > 0$ is a fixed small number chosen as in Section 3.

We first verify the second condition of Lemma 6.6. Given $u \in \mathfrak{M}$, the definition of L_λ and the estimate (3.6) imply

$$|\xi_0 L_\lambda u|_\varepsilon = |\lambda \xi_T u + (1-\lambda)\, \xi_T \mathscr{H}_0 \xi_0 u|_\varepsilon \leqslant \lambda\, \|u\|_\partial + (1-\lambda)\, e^{-\varepsilon/4\, T}\, \|u\|_\partial,$$
$$|\xi_T L_\lambda u|_\varepsilon = |\lambda \xi_T \mathscr{H}_0 \xi_T u + (1-\lambda)\, \xi_T \mathscr{H}_0 \xi_0 u|_\varepsilon \leqslant e^{-\varepsilon/4\, T}\, \|u\|_\partial.$$

Thus,

$$\|L_\lambda u\|_\partial \leqslant \Gamma_\lambda \|u\|_\partial, \qquad \Gamma_\lambda := \lambda + (1-\lambda)\, e^{-\varepsilon/4\, T}.$$

To verify the first condition of Lemma 6.6 we consider an arbitrary $u \in \operatorname{Im}(L_\lambda)$ with

$$u = \lambda \mathscr{H}_0 \xi_T w + (1-\lambda)\, j \xi_T \mathscr{H}_0 \xi_0 w$$

for some $w \in \mathfrak{M}$. The extreme values $\xi_0 u$ and $\xi_T u$ satisfy

$$\begin{cases} \xi_0 u = \lambda \xi_T w + (1-\lambda)\, \xi_T \mathscr{H}_0 \xi_0 w, \\ \xi_T u = \lambda \xi_T \mathscr{H}_0 \xi_T w + (1-\lambda)\, \xi_T \mathscr{H}_0 \xi_0 w. \end{cases}$$

In particular,

$$\xi_0 u - \xi_T u = \lambda (I - \xi_T \mathscr{H}_0)\, \xi_T w.$$

Since $\xi_T \mathscr{H}_0$ has spectral radius less than 1, we can invert $I - \xi_T \mathscr{H}_0$ to obtain

$$|\xi_T w|_\varepsilon \leqslant \frac{1}{\lambda}\, |(I - \xi_T \mathscr{H}_0)^{-1}|_\varepsilon\, 2\, \|u\|_\partial.$$

Also,

$$(1-\lambda)\, |\xi_T \mathscr{H}_0 \xi_0 w|_\varepsilon \leqslant |\xi_0 u|_\varepsilon + \lambda\, |\xi_T w|_\varepsilon \leqslant (1 + 2\, |(I - \xi_T \mathscr{H}_0)^{-1}|_\varepsilon)\, \|u\|_\partial.$$

The relationship between u and w and the continuity of $\mathscr{H}_0$ and j lead to an estimate of the type

$$k_\lambda \|u\|_{\mathfrak{M}, \varepsilon} \leqslant \|u\|_\partial, \qquad u \in \operatorname{Im}(L_\lambda).$$

The inequality

$$\|u\|_\partial \leqslant \|u\|_{\mathfrak{M}, \varepsilon}$$

is obvious from the definition of the semi-norm. This completes the proof.

LEMMA 6.8. *There exists $r^* > 0$ such that if $r > r^*$ then*

$$\operatorname{Fix}(\lambda \mathscr{A}_2 + (1-\lambda)\,\mathscr{P}_\sharp) \cap \partial G_\sharp(r) = \varnothing$$

for each $\lambda \in [0, 1]$.

Proof. To start with we notice that $\mathscr{A}_2$ satisfies

$$\mathscr{A}_2 = \mathscr{H}_0 \circ \xi_T - \mathscr{H}_0 \circ \xi_0 + \Sigma \circ \xi_0. \tag{6.17}$$

This is a consequence of the Volterra's identity (5.3) and the definition of $\mathscr{A}_2$.

Let u be a fixed point of $\lambda \mathscr{A}_2 + (1-\lambda)\,\mathscr{P}_\sharp$. The previous formula (6.17) together with the definitions of $\mathscr{P}$ and $\mathscr{P}_\sharp$ lead to

$$u = \lambda \mathscr{H}_0 \xi_T u - \lambda \mathscr{H}_0 \xi_0 u + \lambda \Sigma \xi_0 u + (1-\lambda)\; j \xi_T \Sigma \xi_0 u. \tag{6.18}$$

If we apply ξ_0 and ξ_T to this identity we obtain

$$\begin{aligned} \xi_0 u &= \lambda \xi_T u + (1-\lambda)\, \xi_T \Sigma \xi_0 u, \\ \xi_T u &= \lambda \xi_T \mathscr{H}_0 \xi_T u - \lambda \xi_T \mathscr{H}_0 \xi_0 u + \xi_T \Sigma \xi_0 u. \end{aligned} \tag{6.19}$$

Here we have used (3.5) and (6.9).

Multiplying the second equation by $1-\lambda$ and subtracting it from the first, we are lead to

$$\xi_0 u - \xi_T u = \lambda(1-\lambda)\, \xi_T \mathscr{H}_0(\xi_0 u - \xi_T u).$$

Next we are going to deduce from this identity that u is periodic, that is, $\xi_0 u = \xi_T u$. This is obvious for $\lambda = 0$ or $\lambda = 1$ and so we assume $\lambda \in (0, 1)$. The proof of Lemma 3.1 shows that the spectral radius of $\xi_T \mathscr{H}_0$ is strictly less than 1. Thus $\frac{1}{\lambda(1-\lambda)} > 1$ is not an eigenvalue of this operator and so $\xi_0 u - \xi_T u$ must vanish.

Once we know $\xi_0 u = \xi_T u$ we go back to the first equation of (6.19) to deduce that $\xi_0 u$ is a fixed point of $\mathscr{P}$.

We are now in a position to prove the Lemma. Let us assume that u belongs to $\operatorname{Fix}(\lambda \mathscr{A}_2 + (1-\lambda)\,\mathscr{P}_\sharp) \cap \partial G_\sharp(r)$ and try to reach a contradiction for large r. From the previous discussion and the definition of $G_{\#}(r)$ we know that

$$\xi_0 u \in \mathrm{Fix}(\mathscr{P}) \cap \bar{G}.$$

The assumptions on G imply that $\xi_0 u$ cannot lie on ∂G. In these circumstances $u \in \partial G_{\#}(r)$ implies

$$\|u\|_{\mathfrak{M}} = r. \tag{6.20}$$

Since $cl_w(G) \subset \mathfrak{D}$ the following number is finite,

$$\rho := \sup \{ \|\Sigma(u_0, v_0)\|_{\mathfrak{M}} / (u_0, v_0) \in G \}.$$

From (6.18) we obtain a bound for u, namely

$$\|u\|_{\mathfrak{M}} \leqslant \rho(1 + \|j\|). \tag{6.22}$$

Define $r^* := \rho(1 + \|j\|)$. Then (6.20) and (6.21) are not compatible when $r > r^*$.

After these two lemmas we can conclude that

$$\deg(I - \mathscr{A}_2, G_{\#}(r); \mathfrak{M}) = \deg(I - \mathscr{P}_{\#}, G_{\#}(r); \mathfrak{M}). \tag{6.22}$$

We know by Lemma 6.5 that $\mathscr{A}_2$ and $\mathscr{F}$ have the same fixed points. Since G and Ω_T have a common core we can deduce (for large r)

$$\mathrm{Fix}(\mathscr{A}_2) \cap G_{\#}(r) = \mathrm{Fix}(\mathscr{F}_T) \cap \Omega_T = \mathrm{Fix}(\mathscr{A}_2) \cap \omega.$$

We have used (6.5). In consequence

$$\deg(I - \mathscr{A}_2, G_{\#}(r); \mathfrak{M}) = \deg(I - \mathscr{A}_2, \omega; \mathfrak{M}) \tag{6.23}$$

and the theorem follows by a chain of identities. Namely, (6.8), (6.16), (6.15), (6.23), (6.22), (6.12) and (6.10).

7. PERIODIC SOLUTIONS OF THE FORCED SINE-GORDON EQUATION

Given $p \in L^1(\mathbb{T}^2)$ we consider the equation

$$\mathfrak{u}_{tt} - \mathfrak{u}_{xx} + c\mathfrak{u}_t + a \sin \mathfrak{u} = p(t, x), \tag{7.1}$$

where c and a are positive numbers. We search for doubly periodic solutions of period 2π; these are solutions satisfying

$$\mathfrak{u}(t+2\pi, x)=\mathfrak{u}(t, x+2\pi)=\mathfrak{u}(t, x) \qquad \text{for all} \quad (t, x)\in\mathbb{R}^2.$$

Given a solution $\mathfrak{u}$ of this type, also the translate $\mathfrak{u}+2\pi$ is a solution. We shall say that two solutions are *geometrically different* if they do not differ by a multiple of 2π.

We plan to apply the abstract framework of Section 4 to this equation (just as in example 4.1). At first sight it seems that this is not possible because p is only in L^1. To overcome this trouble we shall perform a well known change of variables. First we decompose p in the form

$$p(t, x)=\tilde{p}(t, x)+s,$$

where $\tilde{p}$ satisfies $\int_{\mathbb{T}^2}\tilde{p}=0$ and $s\in\mathbb{R}$ is the average of p. Next we find the unique function $P\in C(\mathbb{T}^2)$ which is a solution (in the sense of distributions) of the linear problem

$$P_{tt}-P_{xx}+cP_t=\tilde{p}(t, x) \qquad \text{in } \mathfrak{D}'(\mathbb{T}^2), \qquad \int_{\mathbb{T}^2} P=0 \tag{7.2}$$

(see Proposition 4.4. in [18]). The change of variables

$$\mathfrak{u}=u+P(t, x)$$

transform (7.1) into

$$u_{tt}-u_{xx}+cu_t+a\sin(u+P(t, x))=s. \tag{7.3}$$

This equation fits in the class discussed in example 4.1 From now on we shall always stay in the setting introduced by examples 3.2 and 4.1 (with dimension $n=1$). In particular,

$$H=L^2(\mathbb{T}^1), \qquad V=H^1(\mathbb{T}^1).$$

We shall assume that the time period is

$$T=2\pi$$

and so a doubly periodic solution of (7.3) will be understood as a solution of the periodic boundary problem (4.1)–(5.1), where (4.1) is now the abstract version of (7.3).

The only property of the sine function that will be employed in the rest of the Section is the periodicity. For this reason it seems convenient to replace (7.3) by the more general equation

$$u_{tt} - u_{xx} + cu_t + \Phi(t, x, u) = s, \tag{7.4}$$

where Φ is a function 2π-periodic in each variable. It will be assumed that Φ is in the class $C^{0,1}(\mathbb{T}^2 \times \mathbb{T}^1)$ and $s \in \mathbb{R}$ will act as a parameter. The notion of doubly periodic solution previously introduced for (7.3) also applies for this equation.

In the previous paper [18] we already studied the doubly periodic solutions of (7.4) under the assumption

$$\frac{\partial \Phi}{\partial u}(t, x, u) \leqslant \nu \qquad \text{for all} \quad (t, x, u) \in \mathbb{R}^2 \times \mathbb{R}. \tag{7.5}$$

The number $\nu = \nu(c)$ was introduced in [18]. It determines the region of parameters where it is possible to find a maximum principle for the telegraph equation. We cannot compute it but some properties of ν were found in [18]. In particular,

$$\frac{c^2}{4} < \nu \leqslant \frac{c^2}{4} + \frac{1}{4}.$$

We shall combine the results in Section 6 with [18] to obtain the following result.

THEOREM 7.1. *Assume that* (7.5) *holds. Then there exists an interval* $I_\Phi = [s_-, s_+]$, $s_- \leqslant s_+$, *such that* (7.4) *has*

- *no doubly periodic solution if* $s < s_-$ *or* $s > s_+$;
- *at least one doubly periodic solution if* $s = s_-$ *or* s_+;
- *at least two doubly periodic solutions* (*geometrically different*) *if* $s_- < s < s_+$.

Moreover, at least one of these solutions is not asymptotically stable when $s_- < s < s_+$ *and none of the doubly periodic solutions can be asymptotically stable when* $s = s_-$ *or* s_+.

Remarks. 1. This theorem is inspired by well known results for ordinary differential equations of pendulum-type (see [13] and [17]). In that case a restriction like (7.5) was not required.

2. The existence of an interval I_{Φ} such that (7.4) has a doubly periodic solution whenever $s \in I_{\Phi}$ was already proved in [18]. The result on multiplicity and the information on stability properties seem to be new.

3. In some physical situations it may be interesting to look for doubly periodic solutions of the second kind. The terminology is taken from [14]. These are solutions satisfying

$$u(t+2\pi, x) = u(t, x) + 2N\pi, \qquad u(t, x+2\pi) = u(t, x)$$

for some integer $N \neq 0$. The study of this class of solutions (for fixed N) is reduced to the case $N=0$ by means of the change of unknown

$$v(t, x) = u(t, x) - Nt.$$

The rest of this Section will be devoted to prove Theorem 7.1 First of all we want to apply the results in [18] to justify the existence of the interval I_{Φ}. To do this we must be cautious: the concept of doubly periodic solution employed in [18] does not coincide with the notion induced by the abstract framework of Section 4. In [18] we worked with functions in $C(\mathbb{T}^2)$ satisfying (7.4) in the sense of distributions; in the present paper we work with functions in $\mathfrak{M}_T$ solving (4.1). It is easy to prove that a solution of (4.1)-(5.1) is also a solution in the sense of [18]. The key fact is the immersion of $\mathfrak{M}_T$ in $C(\mathbb{T}^2)$. The converse is slightly more delicate and depends upon the following regularity result for the linear equation (7.2): the solution P belongs to $C^1(\mathbb{T}^2)$ if $\tilde{p}$ is in $C(\mathbb{T}^2)$. This can be proven in many ways; for instance, one can use the Green's function of Lemma 5.2. in [18]. Now, if $u \in C(\mathbb{T}^2)$ is a solution of (7.4) in the sense of distributions, then $\tilde{u} = u - \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} u$ is a solution of (7.2) for an appropriate $\tilde{p}$ which belongs to $C(\mathbb{T}^2)$. Thus u is in $C^1(\mathbb{T}^2)$ and, since this space can be immersed in $\mathfrak{M}_T$, the equivalence between the two definitions is easily obtained.

We are now free to employ the results in [18]. In particular the Theorem 4.3. of that paper guarantees the existence of an interval $I_{\Phi} = [s_-, s_+]$ such that (7.4) has doubly periodic solutions if and only if $s_- \leqslant s \leqslant s_+$. It must be noticed that the class of admissible nonlinearities Φ in that theorem was more restrictive than the class considered now in Theorem 7.1. However this difficulty is only formal because the proof in the general case is the same.

Continuation of the proof of Theorem 7.1 (Multiplicity and instability properties). We consider first the case $s = s_-$ or s_+ and prove that doubly periodic solutions are not asymptotically stable. Let u be one of these solutions and let us assume that it is isolated (otherwise the conclusion is obvious). If the index $\gamma_{\mathscr{P}}(u)$ were different from zero then the properties of the degree and the continuous dependence of (7.4) with respect to the parameter s would imply that (7.4) must have a periodic solution for any s close to s_- or to s_+. This is against the definition of the interval I_Φ and therefore we conclude that $\gamma_{\mathscr{P}}(u)=0$. Now the conclusion follows from Corollary 6.2.

Let us now consider the case $s \in (s_-, s_+)$. We want to prove that there are two geometrically different periodic solutions and also that at least one of them is not asymptotically stable. We can assume that all periodic solutions are isolated because otherwise the result is obviously true. We shall prove two preliminary results.

Claim 1. Given $u \in \mathfrak{M}_T$ solution of (7.4) then

$$\gamma_{\mathscr{F}}(u)=\gamma_{\mathscr{F}}(u+2\pi).$$

Claim 2. There exists $u_1, u_2 \in \mathfrak{M}_T$ solutions of (7.4) with

$$\gamma_{\mathscr{F}}(u_1) \neq \gamma_{\mathscr{F}}(u_2).$$

Once we accept these two claims the proof of the Theorem is easily completed. It is clear that the two solutions u_1 and u_2 are geometrically different and at least one of them must have index $\gamma_{\mathscr{F}}(u_i)$ different from one. In view of Corollary 6.2 we can say that this solution is not asymptotically stable.

Before proving the claims we go back to the abstract setting introduced by examples 3.2 and 4.1 The number $\lambda>0$ in the definition of the bilinear form $a(u, v)$ was arbitrary and so we can select $\lambda=\nu$. The Maximum Principle obtained in [18] implies that the linear operator $\mathscr{G}_T$ of Lemma 3.1 is strongly positive. This is understood in the following sense: given a nontrivial function $p \in C([0, T], H)$ with

$$p(t, x) \geqslant 0 \qquad \text{a.e.} \quad x \in \mathbb{R}$$

for each $t \in [0, T]$ then $w=\mathscr{G}_T p$ satisfies

$$w(t, x)>0 \qquad \text{for each} \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Also we know that $\mathscr{G}_T$ is continuous from $C([0, T], H)$ into $\mathfrak{M}_T$ and its norm will be denoted by $\|\mathscr{G}_T\|$.

The definition of $\mathscr{F}_T$ can be made more precise now. Given $u \in \mathfrak{M}_T$, the image $w=\mathscr{F}_T(u)$ is the solution in $\mathfrak{M}_T$ of

$$w_{tt}-w_{xx}+cw_t+\nu w=F(t, x, u(t, x)), \tag{7.6}$$

where

$$F(t, x, u) := s - \Phi(t, x, u) + vu.$$

To prove the first claim we notice that $\mathscr{F}_T$ commutes with the translation operator

$$\mathscr{T}: \mathfrak{M}_T \to \mathfrak{M}_T, \qquad u \mapsto u + 2\pi.$$

The Commutativity Theorem for Leray–Schauder degree and the identity $\mathscr{F}_T = \mathscr{T}^{-1}\mathscr{F}_T\mathscr{T}$ finish the proof.

Let us now pass to the second claim. We can choose functions u_1, u_- and u_+ in $\mathfrak{M}_T$ such that u_1 is a solution of (7.4) and $u_\pm$ are solutions of

$$u_{tt} - u_{xx} + cu_t + \Phi(t, x, u) = s_\pm.$$

Adding or subtracting multiples of 2π we can also assume that these functions satisfy the inequalities

$$u_- < u_1 - 2\pi < u_1 + 2\pi < u_+.$$

Define

$$\Omega_T = \{u \in \mathfrak{M}_T / u_-(t, x) < u(t, x) < u_+(t, x) \; \forall (t, x), \; \|u\|_{\mathfrak{M}_T} < R\},$$

where R is any number satisfying

$$R > \|\mathscr{G}_T\| \; M\sqrt{2\pi}$$

and

$$M := \max\{|F(t, x, u)| / (t, x) \in [0, T] \times \mathbb{R}, \; u_-(t, x) \leqslant u \leqslant u_+(t, x)\}.$$

The set Ω_T has the following *property*: given $u \in \mathfrak{M}_T$ with $u_- \leqslant u \leqslant u_+$ then $w = \mathscr{F}_T u$ belongs to Ω_T.

Let us prove first $w < u_+$ (the inequality $w > u_-$ is similar). The difference $d = u_+ - w$ satisfies

$$d_{tt} - d_{xx} + cd_t + vd = F(t, x, u_+) - F(t, x, u) + s_+ - s.$$

Now it is time to use the condition (7.5) that tells us that the function F is monotone nondecreasing. It implies that the right hand side of the last equation is positive and the positivity of $\mathscr{G}_T$ leads to the conclusion.

To finish the proof of the property we must show that $\|w\|_{\mathfrak{M}_T} < R$. The definition of $\mathscr{F}_T$ implies that

$$w = \mathscr{G}_T(F(\cdot, \; \cdot, u(\cdot))).$$

From here we obtain the estimates

$$\|w\|_{\mathfrak{M}_T} \leqslant \|\mathscr{G}_T\| \ \ \|F(\cdot,\ \cdot, u(\cdot))\|_{C([0,\,T],\,H)} \leqslant \|\mathscr{G}_T\|\ M\sqrt{2\pi}.$$

We are now in a position to prove the second claim. From the previous property we can deduce that

$$\mathscr{F}_T(\bar{\Omega}_T) \subset \Omega_T.$$

Since Ω_T is a bounded and open convex subset of $\mathfrak{M}_T$ we obtain

$$\deg(I - \mathscr{F}_T, \Omega_T; \mathfrak{M}_T) = 1.$$

On the other hand the additivity of degree implies that

$$\deg(I - \mathscr{F}_T, \Omega_T; \mathfrak{M}_T) = \sum_{u \in \mathrm{Fix}(\mathscr{F}_T) \cap \Omega_T} \gamma_{\mathscr{F}}(u).$$

The set $\mathrm{Fix}(\mathscr{F}_T) \cap \Omega_T$ is finite and has at least three elements (u_1, $u_1 + 2\pi$ and $u_1 - 2\pi$ belong to it). In these circumstances the previous identities for the degree of $I - \mathscr{F}_T$ are not compatible unless the claim holds.

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